Points, Vectors, Covectors, and Scalars

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Math 53]

Moreso than in other aspects of multivariable calculus, in vector calculus it is very important to distinguish between points and vectors to understand what is going on.

point notations in $\mathbb{R}^3$: $(1,2,3) = \begin{pmatrix}1\\2\\3\end{pmatrix}$,

$\underset{\bullet}{x} = (x,y,z) = \begin{pmatrix}x\\y\\z\end{pmatrix}$, $\underset{\bullet}{x_0} = (x_0,y_0,z_0) = \begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}$.

vector notations in $\mathbb{R}^3$: $\underset{\to}{i} = \langle 1,0,0\rangle = \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle$, $\underset{\to}{j} = \langle 0,1,0\rangle = \left\langle\begin{matrix}0\\1\\0\end{matrix}\right\rangle$,

$\underset{\to}{k} = \langle 0,0,1\rangle = \left\langle\begin{matrix}0\\0\\1\end{matrix}\right\rangle$, $\langle 1,2,3\rangle = \left\langle\begin{matrix}1\\2\\3\end{matrix}\right\rangle = \underset{\to}{i} + 2\underset{\to}{j} + 3\underset{\to}{k}$,

$\underset{\to}{x} = \langle x,y,z\rangle = \left\langle\begin{matrix}x\\y\\z\end{matrix}\right\rangle = x\underset{\to}{i} + y\underset{\to}{j} + z\underset{\to}{k}$, $\underset{\to}{x_0}$ = etc.

I will write "$\underset{\bullet}{\mathbb{R}^3}$" or "$\underset{\to}{\mathbb{R}^3}$", respectively, when thinking of $\mathbb{R}^3$ as consisting of points or vectors.

A covector "on" $\mathbb{R}^3$ is just a *not translated!* linear function $\underset{\to}{\mathbb{R}^3} \to \mathbb{R}^1$, i.e. a linear function with vectors (not points!) as inputs and scalars as outputs.

(analogous definitions/notations are used in $\mathbb{R}^2, \mathbb{R}^4, \ldots$)

Using the dot product, we can denote any covector by $\boxed{\vec{u}\,\bullet}$ where $\vec{u} \in \vec{\mathbb{R}}^3$ is a vector, thinking of $\boxed{\vec{x}}$ as an input in the expression "$\vec{u}\bullet\vec{x}$"

Example: If $\alpha : \vec{\mathbb{R}}^3 \to \mathbb{R}^1$ is a covector, say $\alpha(\vec{x}) = 2x+3y+4z$, then we may as well just write $\alpha = \boxed{\langle 2,3,4\rangle \bullet}$ instead.

The age old question: What are dx, dy, and dz, really?

Answer: Covectors! $dx = \vec{i}\,\bullet$, $dy = \vec{j}\,\bullet$, and $dz = \vec{k}\,\bullet$.

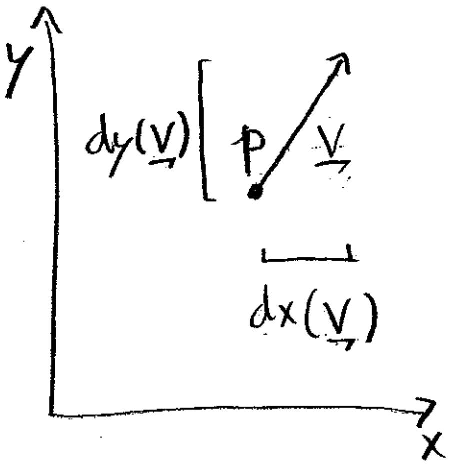
<dramatic pause>

"What?! I thought dx was like an infinitessimal change in x!"

<gaping bewilderment>

Yes, it can still mean that! Let me explain...

... Imagine the point $p \in \mathbb{R}^2$ changes by a tiny amount $\vec{v} \in \mathbb{R}^2$, say $\vec{v} = \langle \frac{1}{100}, \frac{2}{100} \rangle$:



Then $dx(\vec{v}) = \vec{i} \cdot \vec{v}$

$= \langle 1, 0 \rangle \cdot \langle \frac{1}{100}, \frac{2}{100} \rangle = \boxed{\frac{1}{100}}$,

which is how much x changes!

"So dx will have a different value when a different change $\vec{v}$ happens?" Yes. So think of dx as something which tells you how much x changes when a (usually) tiny change happens.

Examples: $dy(\langle \frac{1}{10}, \frac{1}{20}, \frac{1}{30} \rangle) = \boxed{\frac{1}{20}}$

$(dx + dz)(\langle \frac{1}{10}, \frac{1}{20}, \frac{1}{30} \rangle) = \boxed{\frac{4}{30}}$

$(dx + 2dy + 3dz)(\langle \frac{1}{10}, \frac{1}{20}, \frac{1}{30} \rangle) = \boxed{\frac{3}{10}}$

In summary, we have the following

Covector notations "on" $\mathbb{R}^3$: $\underset{\sim}{i}\,\bullet = dx$, $\underset{\sim}{j}\,\bullet = dy$, $\underset{\sim}{k}\,\bullet = dz$,

$$\langle 1,2,3\rangle\bullet = \left\langle \begin{matrix}1\\2\\3\end{matrix}\right\rangle\bullet = \underset{\sim}{i}\,\bullet + 2\underset{\sim}{j}\,\bullet + 3\underset{\sim}{k}\,\bullet = dx + 2dy + 3dz\,.$$